J. Math. Anal. Appl. 365 (2010) 167–170

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A halfspace theorem for mean curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Barbara Nelli[a], Ricardo Sa Earp[b,*]

[a] Universitá di L'Aquila, Italy
[b] Pontificia Universidad Catolica, Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 28 April 2009
Available online 14 October 2009
Submitted by R. Gornet

Keywords:
Second order elliptic equation
Maximum principle
Vertical end
Mean curvature
Halfspace theorem

ABSTRACT

We prove a vertical halfspace theorem for surfaces with constant mean curvature $H = \frac{1}{2}$, properly immersed in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. The proof is a geometric application of the classical maximum principle for second order elliptic PDE, using the family of noncompact rotational $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1. Introduction

D. Hoffman and W. Meeks proved a beautiful theorem on minimal surfaces, the so-called "Halfspace Theorem" in [4]: there is no nonplanar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. In this paper, we focus on complete surfaces with constant mean curvature $H = \frac{1}{2}$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. In the context of $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$, it is natural to investigate halfspace type results.

Before stating our result we would like to emphasize that, in last years there has been work on constant mean curvature surfaces in homogeneous 3-manifolds, in particular in the product space $\mathbb{H}^2 \times \mathbb{R}$: new examples were produced and many theoretical results as well.

A halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many vertically bounded complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [11]. On the contrary, we are able to prove the following result for $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 1. *Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ cannot be properly immersed in the mean convex side of S.*

In [5], L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for $H = \frac{1}{2}$ surfaces on one side of a horocylinder.

The result in [5] is different in nature from our result because in [5], the "halfspace" is one side of a horocylinder, while for us, the "halfspace" is the mean convex side of a rotational simply connected surface.

The proof of our result is a geometric application of the classical maximum principle to surfaces with mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$.

* Corresponding author.
E-mail addresses: nelli@univaq.it (B. Nelli), earp@mat.puc-rio.br (R. Sa Earp).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.031

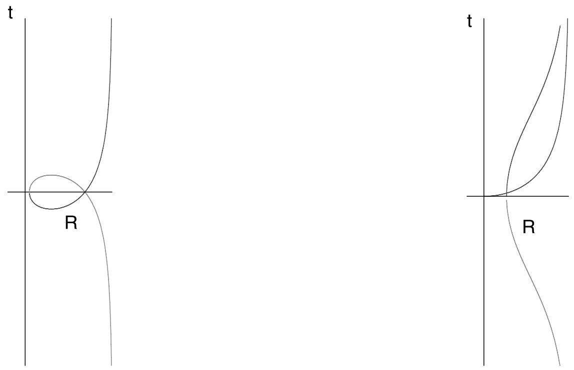

Fig. 1. $H = \frac{1}{2}$: the profile curve in the immersed and embedded case ($R = \tanh \rho/2$).

Maximum Principle. Let S_1 and S_2 be two surfaces with constant mean curvature $H = \frac{1}{2}$ that are tangent at a point $p \in int(S_1) \cap int(S_2)$. Assume that the mean curvature vectors of S_1 and S_2 at p coincide and that, around p, S_1 lies on one side of S_2. Then $S_1 \equiv S_2$. When the intersection point p belongs to the boundary of the surfaces, the result holds as well, provided further that the two boundaries are tangent and both are local graphs over a common neighborhood in $T_pS_1 = T_pS_2$.

The proof of the Maximum Principle is based on the fact that a constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ locally satisfies a second order elliptic PDE (cf. [1–3,6] for the classical proof of the Maximum Principle is in $\mathbb{R}^n$, the proof generalizes to space forms and to $\mathbb{H}^2 \times \mathbb{R}$ as well).

We notice that our surfaces are not compact, while the classical maximum principle applies at a finite point. It will be clear in the proof of Theorem 1 that we are able to reduce the analysis to finite tangent points, because of the geometry of rotational surfaces of mean curvature $H = \frac{1}{2}$.

Our halfspace theorem leads to the following conjecture (strong halfspace theorem).

Conjecture. *Let Σ_1, Σ_2 be two complete properly embedded surfaces with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected surface S of Theorem* 1*. Then Σ_i cannot lie in the mean convex side of Σ_j, $i \neq j$.*

For $H > \frac{1}{\sqrt{2}}$ the conjecture is true and it is known as the maximum principle at infinity (cf. [7]).

2. Vertical halfspace theorem

We recall some properties of rotational surfaces of mean curvature $H = \frac{1}{2}$ that will be crucial in the proof of Theorem 1.

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ in [10]. A careful description of the geometry of these surfaces is contained in Lemma 5.2 and Proposition 5.2 in the Appendix of [8].

For any $\alpha \in \mathbb{R}_+$, there exists a rotational surface $\mathcal{H}_\alpha$ of constant mean curvature $H = \frac{1}{2}$.

For $\alpha \neq 1$, the surface $\mathcal{H}_\alpha$ has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are vertical graphs over the exterior of a disk D_α (see Fig. 1).

By vertical graph we mean the following: the vertical graph of a function u defined on a subset Ω of $\mathbb{H}^2$ is $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$. When the graph has constant mean curvature H, u satisfies the following second order elliptic PDE

$$\mathrm{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 2H \tag{1}$$

where $\mathrm{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2 \times \{0\}$.

Up to vertical translation, one can assume that $\mathcal{H}_\alpha$ is symmetric with respect to the horizontal plane $t = 0$.

For $\alpha = 1$, the surface $\mathcal{H}_1$ has only one end, it is a graph over $\mathbb{H}^2$ and it is denoted by S (second picture in Fig. 1).

When $\alpha > 1$ the surface $\mathcal{H}_\alpha$ is not embedded (first picture in Fig. 1). The self-intersection set is a horizontal circle on the plane $t = 0$. Denote by ρ_α the radius of the intersection circle. For $\alpha < 1$ the surface $\mathcal{H}_\alpha$ is embedded (second picture in Fig. 1).

For any $\alpha \in \mathbb{R}_+$, let $u_\alpha : \mathbb{H}^2 \times \{0\} \setminus D_\alpha \longrightarrow \mathbb{R}$ be the function such that the end of the surface $\mathcal{H}_\alpha$ is the vertical graph of u_α. The asymptotic behavior of u_α has the following form: $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}}$, $\rho \to \infty$, where ρ is the hyperbolic distance from the origin. The positive number $\frac{1}{\sqrt{\alpha}} \in \mathbb{R}_+$ is called the *growth* of the end.

The function u_α is vertical along the boundary of D_α. Furthermore the radius r_α is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \to 0$ or $\alpha \to \infty$. As we pointed out before, the function $u_1 = 2\cosh(\frac{\rho}{2})$ is entire and its graph corresponds to the unique simply connected rotational example S.

Notice that, any end of an immersed rotational surface ($\alpha > 1$) has growth smaller than the growth of S, while any end of an embedded rotational surface ($\alpha < 1$) has growth greater than the growth of S. This means that the intersection between any $\mathcal{H}_\alpha$ and S is a compact set.

Theorem 1 is called "vertical" because the end of the surface Σ is vertical, as it is contained in the mean convex side of S.

Proof of Theorem 1. One can assume that the surface S is tangent to the slice $t = 0$ at the origin and it is contained in $\{t \geqslant 0\}$. Suppose, by contradiction, that Σ is contained in the mean convex side of S. Lift vertically S. If there is an interior contact point between Σ and the translation of S, one has a contradiction by the maximum principle. As Σ is properly immersed, Σ is asymptotic at infinity to a vertical translation of S. One can assume that the surface Σ is asymptotic to the S tangent to the slice $t = 0$ at the origin and contained in $\{t \geqslant 0\}$.

Let h be the height of one lowest point of Σ, i.e. $h = \min\{t \mid (x, y, t) \in \Sigma\}$. Denote by $S(h)$ the vertical lifting of S of length h. One has one of the following facts.

- $S(h)$ and Σ have a first finite contact point p: this means that $S(h - \varepsilon)$ does not meet Σ at a finite point, for $\varepsilon > 0$ and then $S(h)$ and Σ are tangent at p with mean curvature vector pointing in the same direction. In this case, by the maximum principle $S(h)$ and Σ should coincide. Contradiction.
- $S(h)$ and Σ meet at a point p, but p is not a first contact point. Then, for ε small enough, $S(h - \varepsilon)$ intersects Σ transversally.

Denote by W the noncompact subset of $\mathbb{H}^2 \times \mathbb{R}$ above S and below $S(h - \varepsilon)$.

It follows from the maximum principle that there are no compact components of Σ contained in W. Denote by Σ_1 a noncompact connected component of Σ contained in W. By definition of Σ_1, the boundary $\partial\Sigma_1$ is contained in $\partial W \setminus S = S(h - \varepsilon)$. Consider the family of rotational nonembedded surfaces $\mathcal{H}_\alpha$, $\alpha > 1$. Translate each $\mathcal{H}_\alpha$ vertically in order to have the waist on the plane $t = h - \varepsilon$. By abuse of notation, we continue to call the translation, $\mathcal{H}_\alpha$. Denote by $\mathcal{H}_\alpha^+$, the part of the surface outside the vertical cylinder of radius ρ_α. Notice that $\mathcal{H}_\alpha^+$ is embedded and it is a vertical graph. When $\alpha \longrightarrow \infty$, then $\rho_\alpha \longrightarrow \infty$ as well. Furthermore the growth of the end of $\mathcal{H}_\alpha^+$ is smaller than the growth of S. Hence when α is great enough, say α_0, $\mathcal{H}_{\alpha_0}^+$ is outside the mean convex side of S. Then, $\mathcal{H}_{\alpha_0}^+$ does not intersect Σ. Furthermore, when $\alpha \longrightarrow 1$, $\mathcal{H}_\alpha^+$ converge to $S(h - \varepsilon)$. Now, start to decrease α from α_0 to one. Before reaching $\alpha = 1$, the surface $\mathcal{H}_\alpha^+$ first meets S and then touches Σ_1 tangentially at an interior finite point, with Σ_1 above $\mathcal{H}_\alpha^+$. This depends on the following two facts.

- The boundary of Σ_1 lies on $S(h - \varepsilon)$ and the boundary of any of the $\mathcal{H}_\alpha^+$ lies on the horizontal plane $t = h - \varepsilon$.
- The growth of any of the $\mathcal{H}_\alpha^+$ is strictly smaller than the growth of S. Thus the end of $\mathcal{H}_\alpha^+$ is outside the mean convex side of S.

The existence of such an interior tangency point is a contradiction by the maximum principle. $\square$

Remark 1. Our result is sharp in the following sense. There are examples of complete, $H = \frac{1}{2}$ surfaces properly immersed in the nonmean convex side of the simply connected rotational surface S of Theorem 1. Consider any $\mathcal{H}_\alpha$ with $\alpha > 1$. As the growth of $\mathcal{H}_\alpha$ is less than the growth of S, $\mathcal{H}_\alpha \cap S$ is a compact set. Then, a suitable vertical downward translation of $\mathcal{H}_\alpha$ is contained in the nonmean convex side S.

It is worth noticing that there are many examples of entire graphs of mean curvature $H = \frac{1}{2}$ with a nonvertical end, i.e., with points of the asymptotic boundary at finite height; see, for instance, [9]. A significant example is given by the following graph (halfplane model for $\mathbb{H}^2$):

$$t = \frac{\sqrt{x^2 + y^2}}{y}, \quad y > 0.$$

It has mean curvature $H = \frac{1}{2}$, and its asymptotic boundary contains two vertical half straight lines (see [9, Eq. (31), Fig. (12)]).

Let us now discuss some consequences of Theorem 1.

First we need to recall the following notion. For a given circle C in $\mathbb{H}^2 \times \{0\}$, denote by Z the vertical cylinder over C, that is $Z = \{(x, y, t) \mid (x, y) \in C,\ t \in \mathbb{R}\}$. An end E is *cylindrically bounded* if there exists a vertical cylinder Z such that, up to a reflection about the slice $\{t = 0\}$, E is contained in the mean convex side of $Z \cap \{t \geqslant 0\}$.

Corollary 1. *Let Σ be a complete properly immersed surface with mean curvature $H = \frac{1}{2}$ with cylindrically bounded ends. Then Σ must have more than one end.*

Proof. Assume by contradiction that Σ has only one cylindrically bounded end E. Then there exists a vertical cylinder Z such that E lies in the mean convex side of $Z \cap \{t > 0\}$. In particular, one can choose the cylinder Z such that the whole surface Σ is contained in $Z \cap \{t > 0\}$. It is clear that $Z \cap \{t > 0\}$ is contained in the mean convex side of a suitable vertical translation of the simply connected surface S. Hence Σ is contained in the mean convex side of some vertical translation of S as well. Now Theorem 1 yields $\Sigma = S$ (in fact we only need the first part of the proof of Theorem 1), which is a contradiction, since S is not cylindrically bounded. $\square$

As we remarked before, there are many entire graphs of mean curvature $H = \frac{1}{2}$. The following consequence of Theorem 1 gives some information about their geometry.

Corollary 2. *Let Σ be an entire graph of mean curvature $H = \frac{1}{2}$. If Σ is not rotational, then it intersects the interior of the complement of the mean convex side of S.*

References

[1] J.L. Barbosa, R. Sa Earp, Prescribed mean curvature hypersurfaces in $\mathbb{H}^{n+1}$ with convex planar boundary, II, Semin. Theor. Spectr. Geom. 16 (1998) 43–79.
[2] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1998.
[3] H. Hopf, Differential Geometry in the Large, Lecture Notes in Math., vol. 1000, Springer-Verlag, 1983.
[4] D. Hoffman, W. Meeks III, The strong halfspace theorem for minimal surfaces, Invent. Math. 101 (1) (1990) 373–377.
[5] L. Hauswirth, H. Rosenberg, J. Spruck, On complete mean curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$, Comm. Anal. Geom. 16 (5) (2009) 989–1005.
[6] B. Nelli, A survey on Alexandrov–Bernstein–Hopf theorems, http://www.dmmm.uniroma1.it/~nelli/articoli.html, Mat. Contemp., in press.
[7] B. Nelli, H. Rosenberg, Global properties of constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$, Pacific J. Math. 226 (1) (2006) 137–152.
[8] B. Nelli, R. Sa Earp, W. Santos, E. Toubiana, Uniqueness of H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, $|H| \leqslant \frac{1}{2}$, with boundary one or two parallel horizontal circles, Ann. Global Anal. Geom. 33 (4) (2008) 307–321.
[9] R. Sa Earp, Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, J. Aust. Math. Soc. 85 (2008) 113–143.
[10] R. Sa Earp, E. Toubiana, Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, Illinois J. Math. 49 (4) (2005) 1323–1362.
[11] R. Sa Earp, E. Toubiana, An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$, Math. Ann. 342 (2) (2008) 309–331.